Exact discrete minimization for TV+L0 image decomposition models

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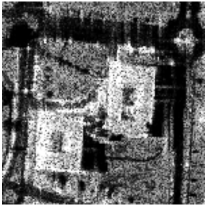
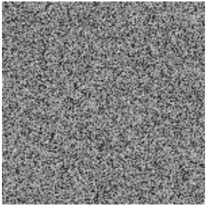
Synthetic aperture radar (SAR) images denoising:



noisy data

$\boldsymbol{v}$

$=$



noise

$\boldsymbol{n}$

$\times$

scene

$\boldsymbol{u}$

noise / signal separation using a variational approach:
recover scene $\boldsymbol{u}$ as the minimizer of $E_{\text{data}}(\boldsymbol{u}, \boldsymbol{v}) + E_{\text{reg}}(\boldsymbol{u})$

Radar scene distinctive feature: *strong scatterers* (very bright dots)

Q: How to model such scenes?
Q: How to compute the corresponding minimizers?

Overview

1. TV+L0 image decomposition models

2. Exact discrete minimization by graph-cuts

3. Results and discussion

1. Denoising and image decomposition

Total variation denoising: $\hat{\boldsymbol{u}} = \arg\min_{\boldsymbol{u}} \; E_{\text{data}}(\boldsymbol{u}, \boldsymbol{v}) + E_{\text{reg}}(\boldsymbol{u})$

$$E_{\text{reg}}(\boldsymbol{u}) = \mathrm{TV}(\boldsymbol{u}) := \int \sqrt{(\nabla_x \boldsymbol{u})^2 + (\nabla_y \boldsymbol{u})^2} \; \mathrm{d}x \, \mathrm{d}y$$

- 👍 preserves sharp boundaries
- 👎 cartoon-like images (staircasing effect), favors larger regions

Image decomposition: e.g., $E_{\text{data}}(\boldsymbol{u} - \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|_1$

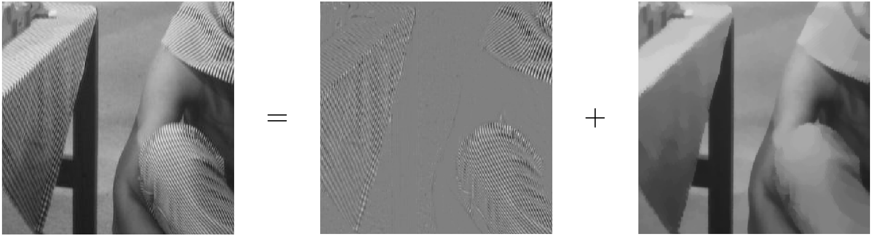
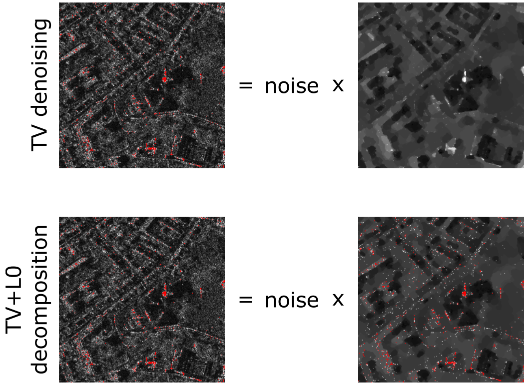


image = texture + geometry

(illustration from [Yin, Goldfarb & Osher 2005])

TV denoising vs TV+L0 image decomposition:



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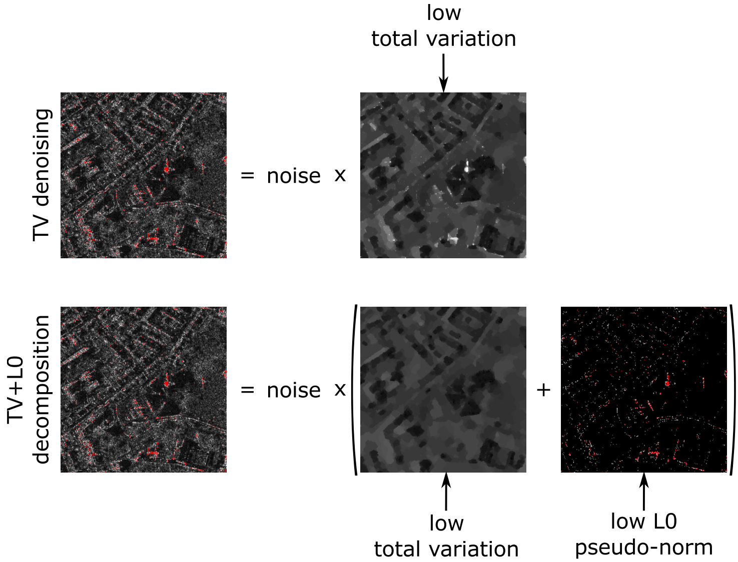

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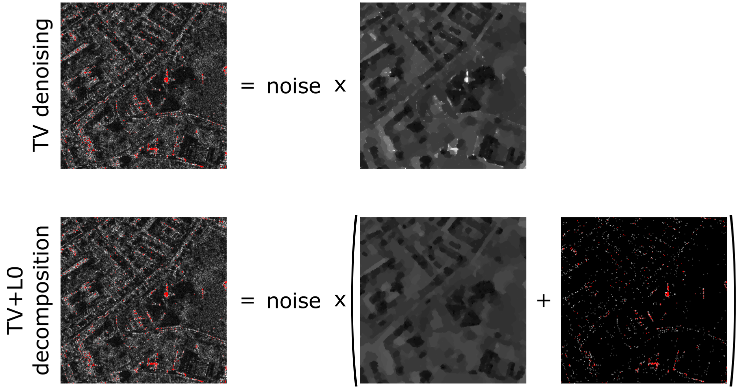

Image decomposition provides a way to enrich scene modeling

2. Energy minimization problem

$$\widehat{(\boldsymbol{u}_{\mathrm{BV}}, \boldsymbol{u}_{\mathrm{S}})} = \underset{(\boldsymbol{u}_{\mathrm{BV}}, \boldsymbol{u}_{\mathrm{S}})}{\arg\min} \quad E_{\mathsf{data}}(\boldsymbol{v}, \boldsymbol{u}_{\mathrm{BV}}, \boldsymbol{u}_{\mathrm{S}}) + E_{\mathsf{reg}}(\boldsymbol{u}_{\mathrm{BV}}, \boldsymbol{u}_{\mathrm{S}})$$

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1. Image formation model

Assumption: *separable* likelihood
(no blurring, uncorrelated noise)

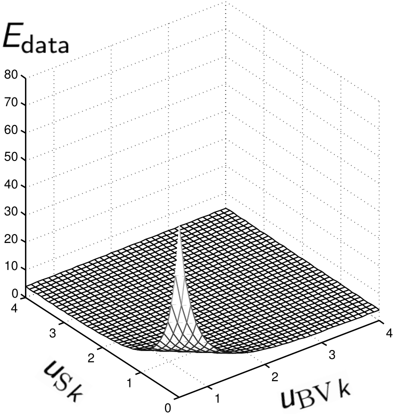
$$E_{\mathrm{data}} = \sum_k -\log \mathrm{p}(v_k | u_{\mathrm{BV}\,k}, u_{\mathrm{S}\,k})$$

Speckle noise → Rayleigh distribution:

$$E_{\mathrm{data}} = \sum_k \frac{{v_k}^2}{(u_{\mathrm{BV}\,k} + u_{\mathrm{S}\,k})^2} + 2\log(u_{\mathrm{BV}\,k} + u_{\mathrm{S}\,k})$$

Positivity constraints:

$$\forall k, \; u_{\mathrm{BV}\,k} > 0 \;\; \text{and} \;\; u_{\mathrm{S}\,k} \geq 0$$



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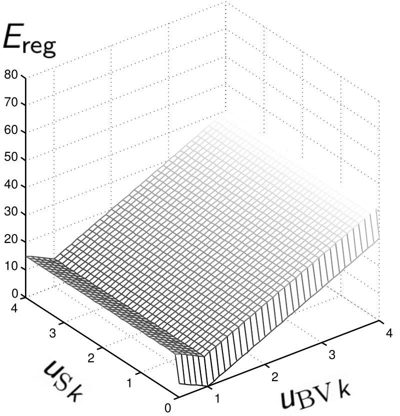
2. Image decomposition model

Prior model:
decomposition into *sparse* and *bounded variations* components

$$E_{\mathsf{reg}} = \beta_{\mathrm{S}}\, \mathrm{L0}(\boldsymbol{u}_{\mathrm{S}}) + \beta_{\mathrm{BV}}\, \mathrm{TV}(\boldsymbol{u}_{\mathrm{BV}})$$



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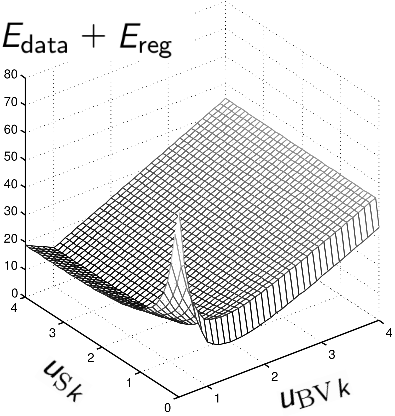
Minimization problem

E_{data} is non-convex (quasi-convex)
E_{reg} is non-convex (due to L0 term)

$\rightarrow$ the problem is non-convex

Variable coupling:
- $\boldsymbol{u}_{\mathrm{BV}}$ and $\boldsymbol{u}_{\mathrm{S}}$ are coupled
- $\boldsymbol{u}_{\mathrm{BV}}$ is spatially coupled

global minimization is hard...



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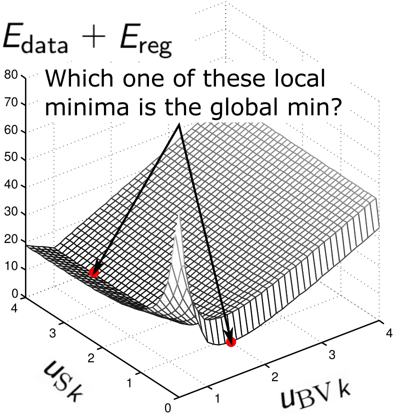

2. Energy minimization problem: reformulation

1. Consider $\boldsymbol{u}_{\mathrm{BV}}$ fixed. The restricted problem is spatially separable:

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   The problem reduces to a 1D problem per pixel (easy).

2. The original problem can be reformulated with $\boldsymbol{u}_{\mathrm{BV}}$ only:

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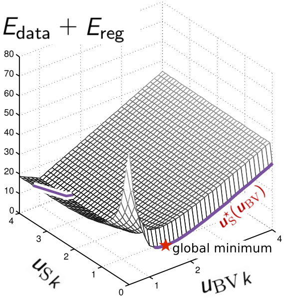

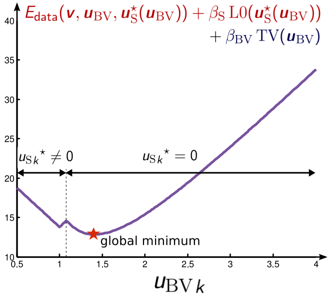

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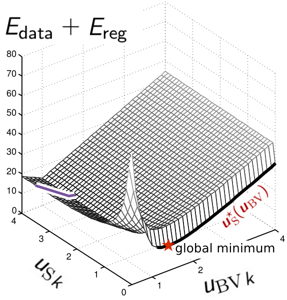

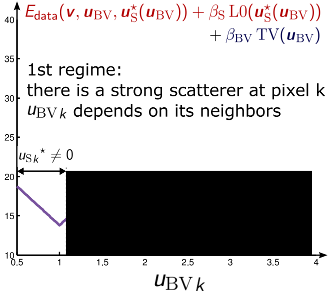

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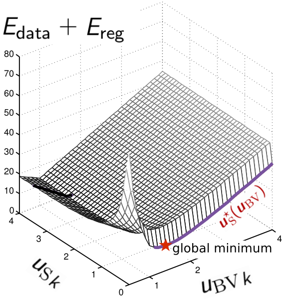

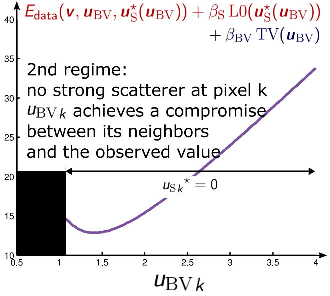

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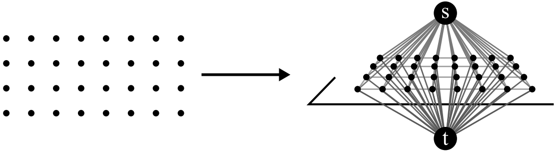
2. Energy minimization problem: graph-cuts methodology

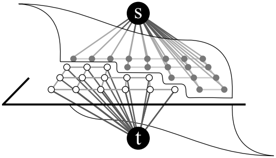
1. The pixel grid is mapped to a graph with two terminal nodes:



pixel grid

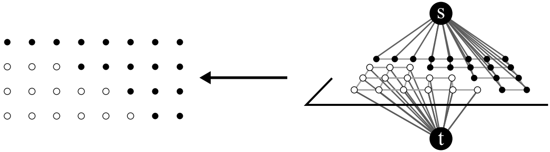
associated graph

2. A minimum s-t-cut is computed:



min s-t-cut

3. The cut is interpreted as a solution of the original problem:

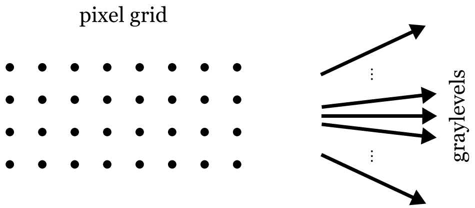


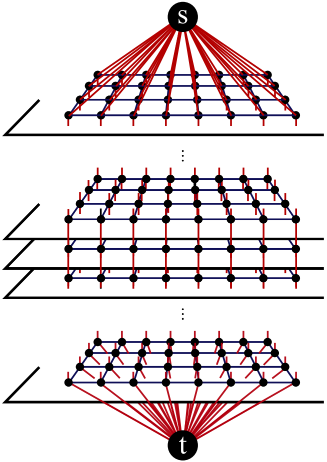
minimizer

min s-t-cut

2. Energy minimization problem: graph construction

$$\underset{\boldsymbol{u}_{\mathrm{BV}}}{\arg\min}\quad E_{\mathsf{data}}(\boldsymbol{v}, \boldsymbol{u}_{\mathrm{BV}}, \boldsymbol{u}^{\star}_{\mathrm{S}}(\boldsymbol{u}_{\mathrm{BV}})) + \beta_{\mathrm{S}}\,\mathrm{L0}(\boldsymbol{u}^{\star}_{\mathrm{S}}(\boldsymbol{u}_{\mathrm{BV}})) + \beta_{\mathrm{BV}}\,\mathrm{TV}(\boldsymbol{u}_{\mathrm{BV}})$$

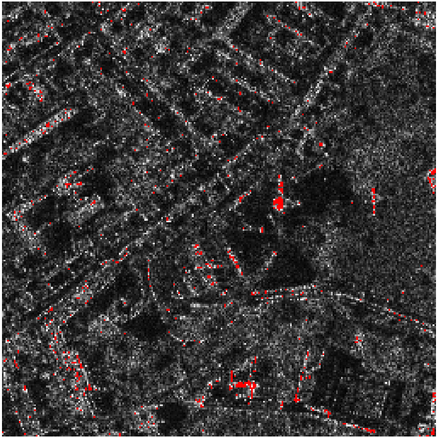




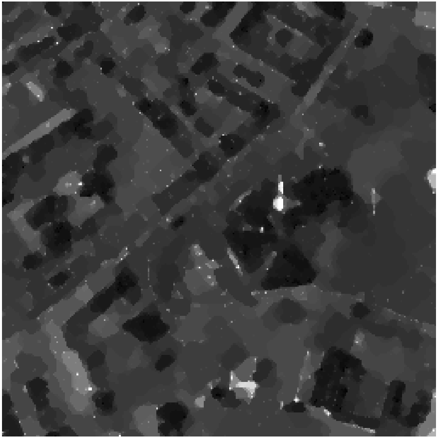
Ishikawa's graph for multi-valued images
[Ishikawa PAMI2003]

- $\boldsymbol{u}_{\mathrm{BV}}$ is decomposed into its level sets
- each level is represented by a layer of the graph
- vertical arcs going downstream represent $E_{\mathsf{data}}(\cdot) + \beta_{\mathrm{S}}\,\mathrm{L0}(\cdot)$
- horizontal arcs represent $\beta_{\mathrm{BV}}\,\mathrm{TV}(\boldsymbol{u}_{\mathrm{BV}})$
- positivity is naturally enforced

3. Results: TV vs TV+L0 decomposition



Noisy data ©CNES/DGA



TV denoising

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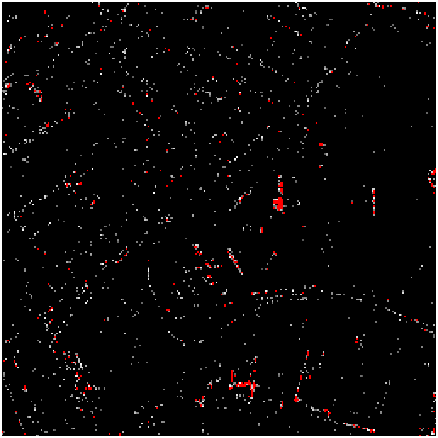
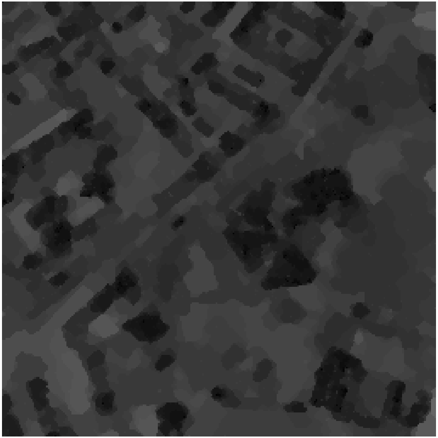


Strong scatterers $\boldsymbol{u}_{\mathrm{S}}$



Homogeneous regions $\boldsymbol{u}_{\mathrm{BV}}$

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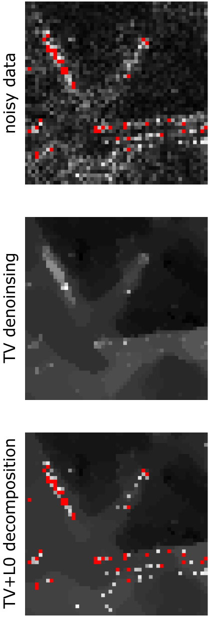

Image decomposition:

- suppresses the bias on strong scatterers (i.e., loss of contrast and suppression of point-like objects)
- better preserves resolution (strong scatterers do not spread)

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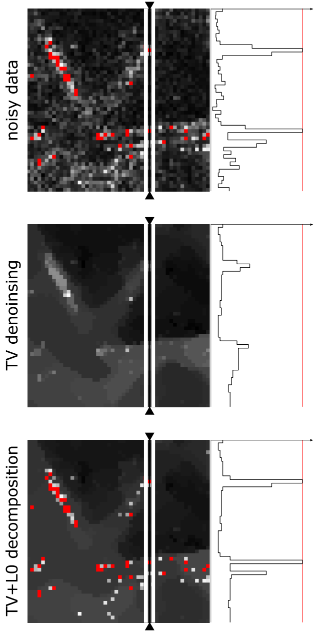

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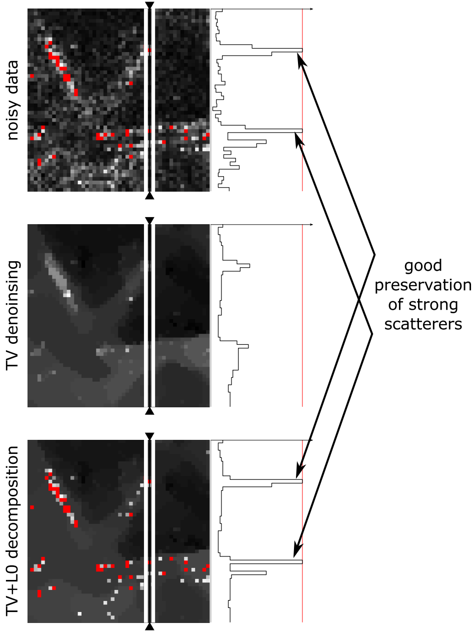

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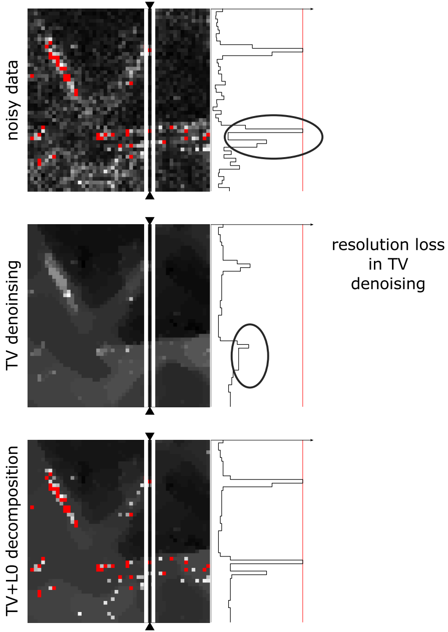

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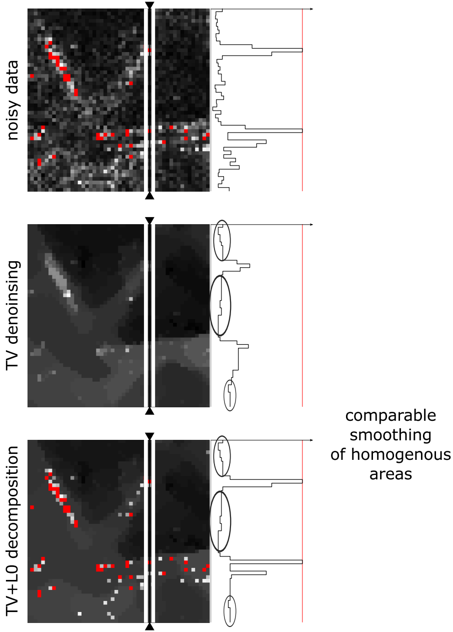

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3. Conclusion

- The prior model benefits from image decomposition
- Decomposition choice: a component with bounded variations $\boldsymbol{u}_{\mathrm{BV}}$ and a sparse component $\boldsymbol{u}_{\mathrm{S}}$
- Minimization of TV+L0 is challenging but exact discrete minimization is possible with graph-cuts
- A drawback of this minimization approach is its memory cost: O(number of pixels $\times$ number of quantization levels)
- More elaborate speckle noise models (strong scatterer + random phasors) can be applied with the proposed decomposition for SAR image denoising ($\rightarrow$ Rice distribution, see paper)

Questions?

loic.denis@cpe.fr

the slides can be downloaded from my homepage
(http://www-obs.univ-lyon1.fr/labo/perso/loic.denis/)