Augmented Lagrangian without alternating directions: practical algorithms for inverse problems in imaging

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Starting point

Many image processing tasks are formulated as a large scale optimization problem:

image restoration

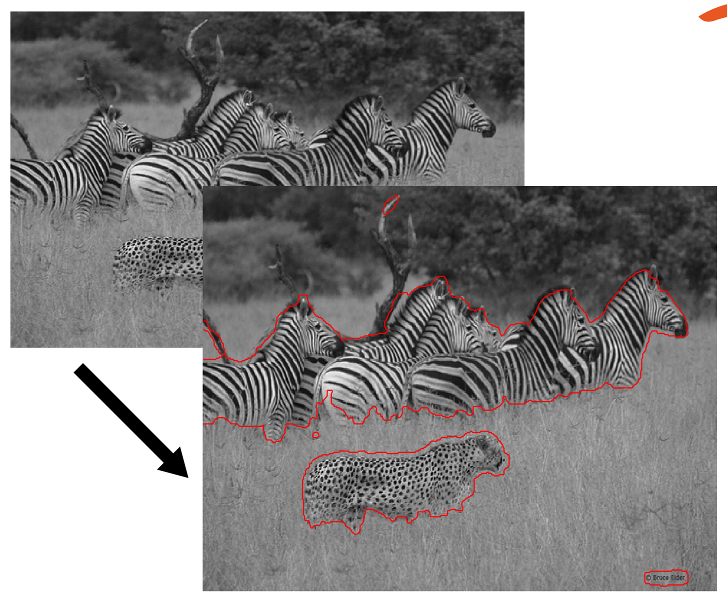


image segmentation

- often requires to tune the regularization parameter
- need fast and reliable optimization methods

⚠ don't want to also tune the optimization algorithm parameters
avoid interplay between regularization & optimization parameter tuning

Typical optimization problems in imaging

$$\arg\min_{\boldsymbol{x}} \; \underbrace{f(\boldsymbol{x})}_{\text{smooth}} + \underbrace{r(\boldsymbol{x})}_{\text{non-smooth}}$$

convex

e.g., L1 norm to promote sparsity or indicator function for constraints

⚠ large scale problems: typically 10^6 to 10^9 unknowns

- 👍 limited-memory quasi-Newton methods are very efficient
- 👎 cannot apply to non-smooth problems
- 👍 many algorithms for non-smooth optimization
- 👎 can be very slow

The augmented Lagrangian reformulation

- the optimization problem is too hard $\Rightarrow$ separate variables

$$\arg\min_{\boldsymbol{x},\boldsymbol{z}} \; f(\boldsymbol{x}) + r(\boldsymbol{z}) \quad \text{subject to} \quad \boldsymbol{x} = \boldsymbol{z}$$

- turn the constrained problem into an unconstrained form with the augmented Lagrangian:

$$\underbrace{f(\boldsymbol{x}) + r(\boldsymbol{z}) + \boldsymbol{u}^{\mathrm{t}}(\boldsymbol{x}-\boldsymbol{z})}_{\text{Lagrangian}} + \underbrace{\frac{\rho}{2}\|\boldsymbol{x}-\boldsymbol{z}\|_2^2}_{\text{augmentation term}}$$

Lagrange multiplier ($\boldsymbol{u}$), optimization parameter (ρ)

Hierarchical optimization

our proposition

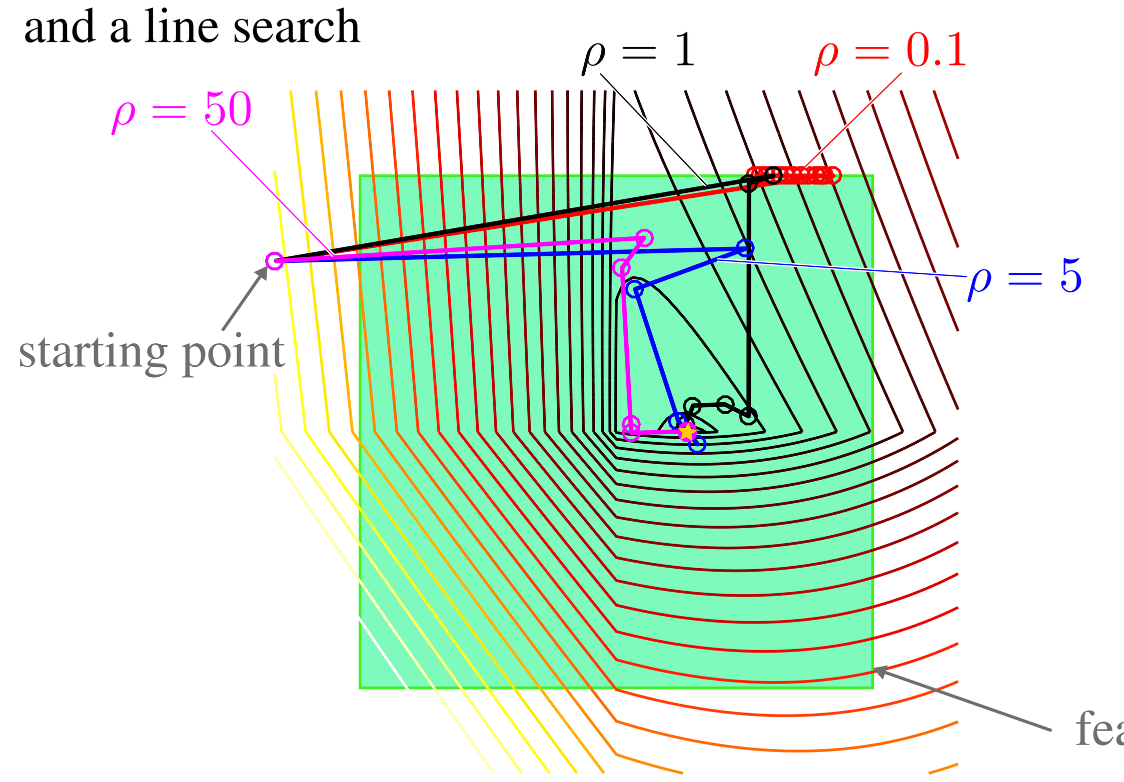
$$\arg\min_{\boldsymbol{x}} \; f(\boldsymbol{x}) + \boldsymbol{u}^{\mathrm{t}}(\boldsymbol{x}-\boldsymbol{z}^\star) + \frac{\rho}{2}\|\boldsymbol{x}-\boldsymbol{z}^\star\|_2^2$$

where $\boldsymbol{z}^\star \equiv \arg\min_{\boldsymbol{z}} \; r(\boldsymbol{z}) + \boldsymbol{u}^{\mathrm{t}}(\boldsymbol{x}-\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{x}-\boldsymbol{z}\|_2^2$

$$\boldsymbol{u} \leftarrow \boldsymbol{u} + \rho(\boldsymbol{x}-\boldsymbol{z})$$

- $\boldsymbol{z}^\star$ can generally be computed in closed-form
- the partially optimized $\boldsymbol{x}$ subproblem is smooth

$\Rightarrow$ can be solved approximately using a quasi-Newton method with $\boldsymbol{\nabla} \equiv \boldsymbol{\nabla} f + \boldsymbol{u} + \rho(\boldsymbol{x}-\boldsymbol{z}^\star)$ and a line search



ADMM

Alternating directions method of multipliers

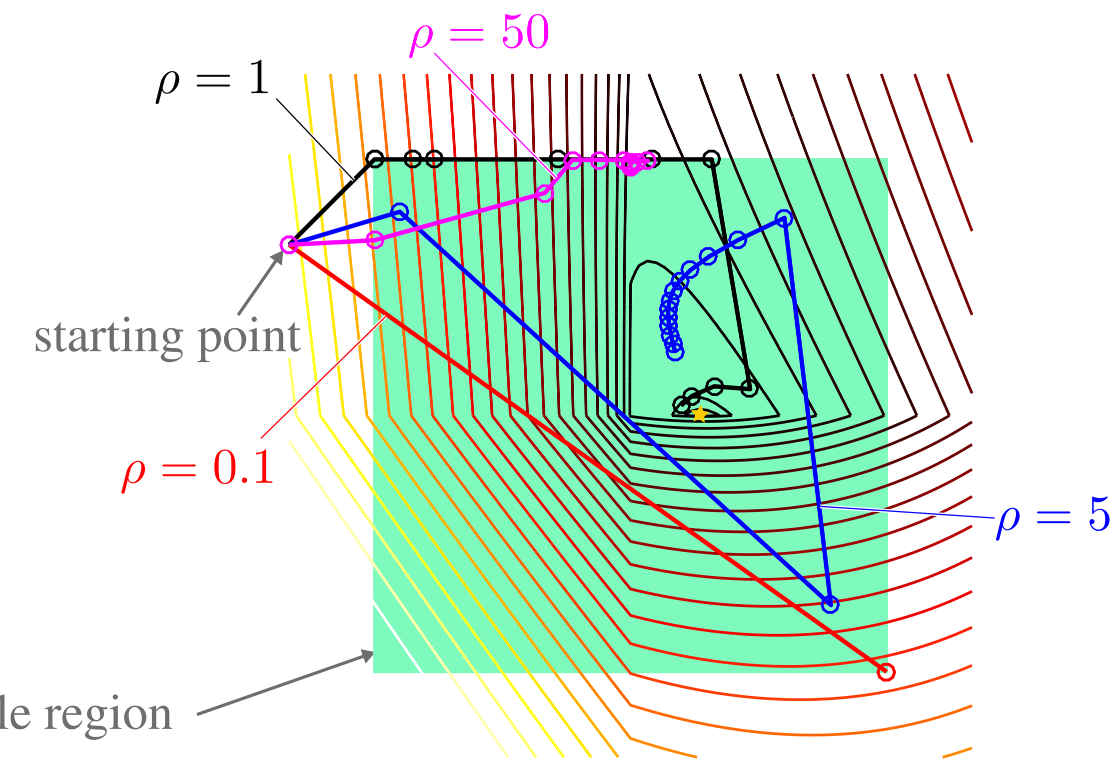
$$\arg\min_{\boldsymbol{x}} \; f(\boldsymbol{x}) + \boldsymbol{u}^{\mathrm{t}}(\boldsymbol{x}-\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{x}-\boldsymbol{z}\|_2^2$$

$$\arg\min_{\boldsymbol{z}} \; r(\boldsymbol{z}) + \boldsymbol{u}^{\mathrm{t}}(\boldsymbol{x}-\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{x}-\boldsymbol{z}\|_2^2$$

$$\boldsymbol{u} \leftarrow \boldsymbol{u} + \rho(\boldsymbol{x}-\boldsymbol{z})$$

- works best with multiple splittings so that each sub-problem can be solved exactly [Matakos, Ramani & Fessler 2013]

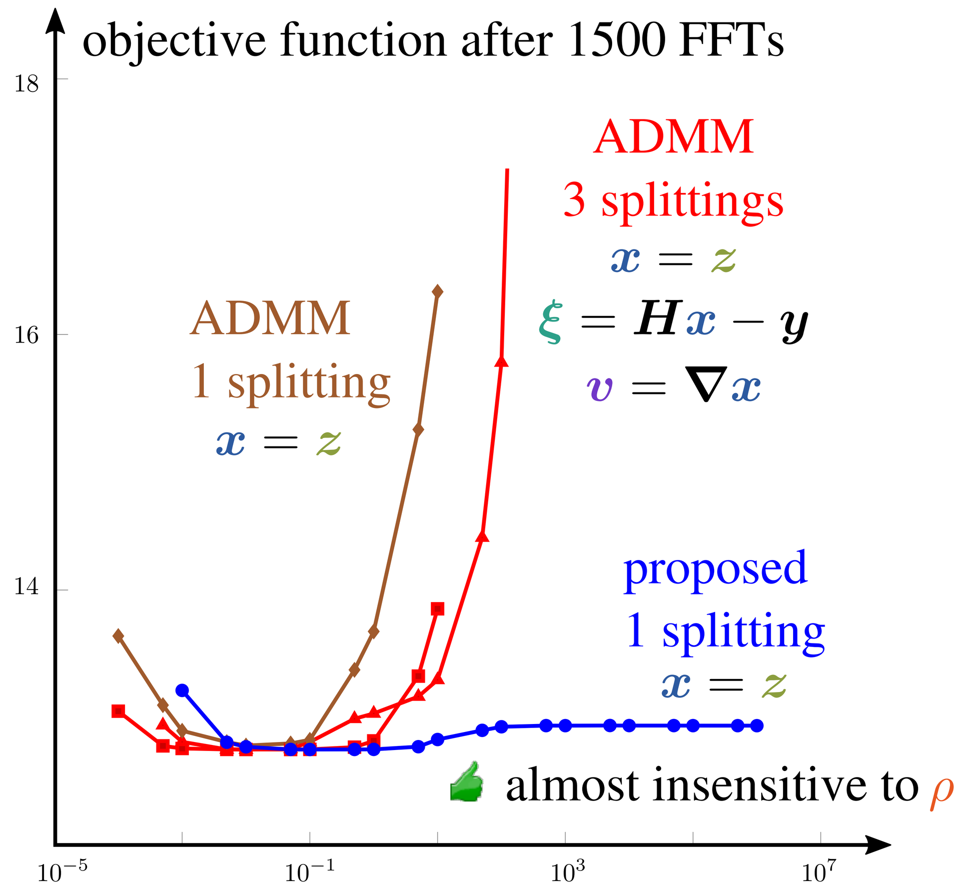
⚠ requires fine-tuning the parameter(s) ρ



Results

- image deconvolution [Matakos, Ramani & Fessler 2013]

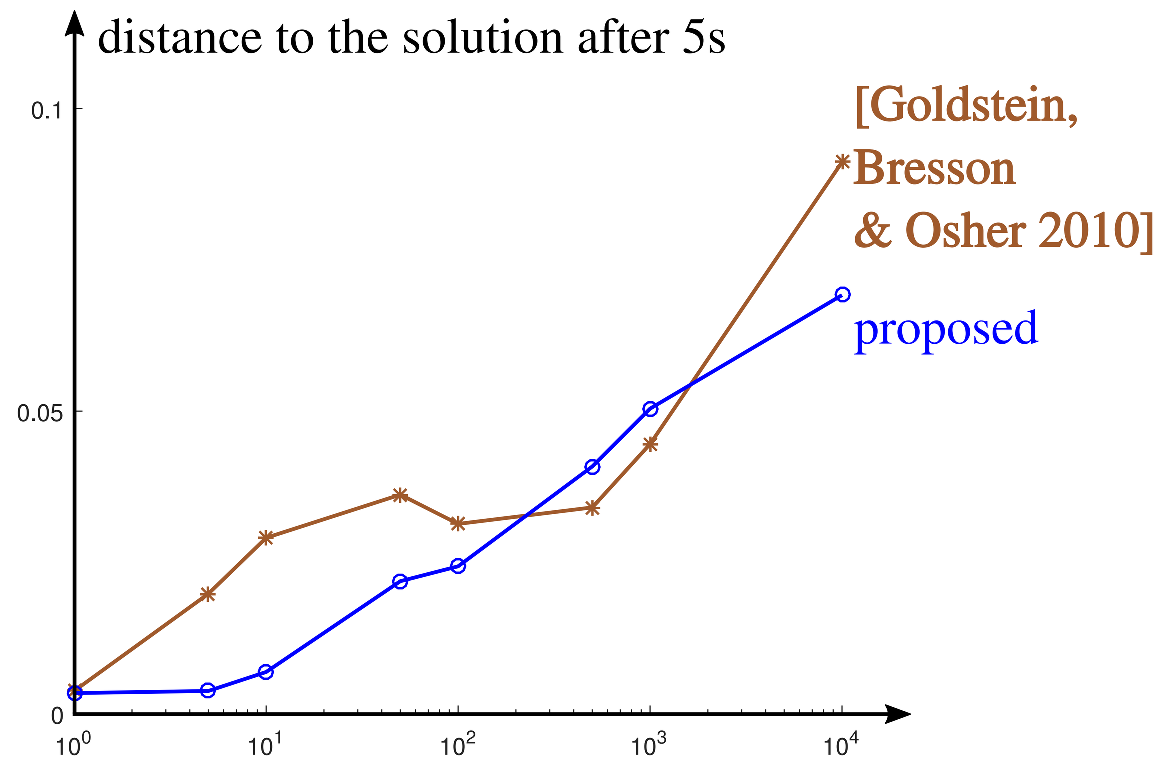
$$\arg\min_{\boldsymbol{x}\geq 0} \; \|\boldsymbol{H}\boldsymbol{x}-\boldsymbol{y}\|_W^2 + \lambda \mathrm{TV}(\boldsymbol{x})$$



- image segmentation [Houhou, Thiran & Bresson 2009] [Goldstein, Bresson & Osher 2010]

$$\arg\min_{0\leq \boldsymbol{x}\leq 1} \; \boldsymbol{h}^{\mathrm{t}}\boldsymbol{x} + \lambda \mathrm{TV}(\boldsymbol{x})$$

with $\boldsymbol{h}$ defined from textural descriptors



➡ reaches state-of-the art performance with **no parameter tuning**

[Matakos, Ramani & Fessler 2013] "Accelerated edge-preserving image restoration without boundary artifacts," IEEE Transactions on Image Processing
[Houhou, Thiran & Bresson 2009] "Fast texture segmentation based on semi-local region descriptor and active contour," Numerical Mathematics: Theory, Methods and Applications
[Goldstein, Bresson & Osher 2010] "Geometric applications of the split Bregman method: segmentation and surface reconstruction," Journal of Scientific Computing

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